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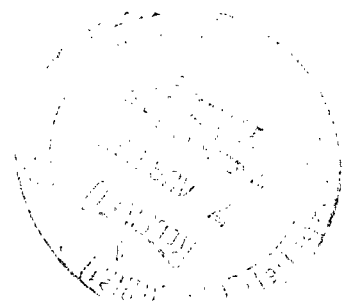
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THEORETICAL SOLUTION OF THE NONLINEAR PROBLEM OF TRANSIENT COOLING OF AN OPAQUE SPHERE IN SPACE

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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ABSTRACT

The temperature distribution $T(r, t)$ in homogeneous opaque spheres with constant radius $r = R$ and constant initial temperature $T_0 = T(r, 0)$ is represented as a closed-form solution of Fourier's equation when heat is radiated from the surface $r = R$ into a vacuum. Substitution of the boundary conditions at $r = 0$ and $r = R$ into this solution yields a system of two nonlinear integro-differential equations for $T(R, t)$ and $T(0, t)$. These equations are solved numerically. Results for $T(r, t)$ are presented for spheres made of quartz with $R = 1$ cm, $R = 15.3$ cm, and $R = \infty$. One application of the solution is to determine the temperature history of meteorite ejecta traveling in space.

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INTRODUCTION

The transient cooling process of spheres with constant radius R and constant material properties is considered under conditions of radiative heat transfer from the surface to the environment and conduction of heat in the material, which is assumed to be homogeneous and opaque to thermal radiation. The particular example which induced this study concerns the temperature history of debris ejected from the moon by a meteorite impact. The spherically symmetric closed-form solution of Fourier's heat conduction equation which is employed here is well known for the case of linear boundary conditions. The mathematical problem is due to the nonlinear boundary condition at $r = R$, following from the Stefan-Boltzmann radiation law. By making minor changes in the derived solution, other forms of this boundary condition may be employed, such as those due to additional convective heat transfer and/or mass transfer. In a forthcoming publication, a generalization of the solution presented here will be employed to solve the nonlinear integro-differential equation, which accounts for both radiative and conductive heat transfer in the material. In the boundary condition at $r = R$, a mass transfer term due to evaporation will be included. Published solutions of Fourier's equation concern linear or linearized forms of the boundary condition at $r = R$ and thus are unreliable when the change of temperature of the radiating surface exceeds a certain small limit.

DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

The problem outlined in the above section is governed by the following relations:

1. Fourier's equation:

$$\frac{\partial T}{\partial t} = \frac{1}{a^2 r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) \quad \text{for } 0 < r < R, \quad t > 0, \quad (1)$$

where $a^2 = \rho c/K$, the thermal diffusivity (Reference 3).

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2. The initial condition at time $t = 0$:

$$T(r, 0) = T_0 = \text{constant for } 0 \leq r \leq R. \quad (2)$$

3. The symmetry condition at the center $r = 0$:

$$\frac{\partial T(0, t)}{\partial r} = 0 \text{ for } t \geq 0. \quad (3)$$

4. The heat balance equation at the surface $r = R$:

$$-K \frac{\partial T(R, t)}{\partial r} = \sigma_0 \epsilon_0 T^4(R, t) \text{ for } t > 0, \quad (4)$$

which equates heat radiated from the surface to heat conducted to the surface.

According to the theory of the parabolic differential equation, the solution is determined by the six quantities T_0 , $T(R, t)$, R , t , $\sigma_0 \epsilon_0 / K$, and $K / \rho c$, which depend on the three fundamental units length, time, and temperature. The π -theorem of dimensional analysis predicts that the solution sought depends on $6 - 3 = 3$ nondimensional power-products of the six quantities mentioned above. By use of the parameter ξ , defined by

$$\xi = \frac{K \rho c}{\pi (\sigma_0 \epsilon_0)^2 T_0^6}, \quad (5)$$

these expressions may be selected as follows:

$$\frac{T(R, t)}{T_0}, \quad \sigma = \frac{t}{\xi}, \quad \text{and} \quad D = \frac{\sqrt{\xi}}{aR} = \frac{K}{R T_0^3 \sigma_0 \epsilon_0 \sqrt{\pi}}. \quad (6)$$

Characteristic features of the solution are shown by ξ the parameter governing the time-scale:

(a) ξ depends on the product of K , ρ , and c , and (b) ξ is very strongly affected by changes of the initial value T_0 of the absolute temperature.

SOLUTION

The second author will discuss in a later publication the existence and the uniqueness of solutions of the parabolic problem as defined by Equations 1-4. It is well known (see Reference 1) that substitution of the transformation

$$T(r, t) = \frac{u(r, t)}{r} + T_0 \quad (7)$$

into the differential equation (Equation 1) yields Fourier's equation

$$\frac{\partial u(r, t)}{\partial t} = \frac{1}{a^2} \frac{\partial^2 u(r, t)}{\partial r^2} \quad (8)$$

for the planar case. The conditions in Equations 2, 3, and 4 are transformed as follows by use of Equation 7:

$$u(r, 0) = 0 \quad \text{for} \quad 0 \leq r \leq R, \quad (9)$$

$$\lim_{r \rightarrow 0} \left(\frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right) = 0 \quad \text{for} \quad t \geq 0, \quad (10)$$

and

$$-K \left(\frac{1}{R} \frac{\partial u(R, t)}{\partial r} - \frac{u(R, t)}{R^2} \right) = \sigma_0 \epsilon_0 \left(T_0 + \frac{u(R, t)}{R} \right)^4 \quad (11)$$

for $t > 0$.

Application of l' Hospital's rule reveals that the left-hand side in Equation 10 tends towards the same limit as $0.5 \partial^2 u / \partial r^2$ as $r \rightarrow 0$. This is equal to $(a^2/2) \partial u(0, t) / \partial t$, according to Equation 8. Therefore, the condition in Equation 10 may be replaced by

$$u(0, t) = 0 \quad \text{for} \quad t \geq 0. \quad (12)$$

By use of the complementary error integral (Reference 1),

$$F(X) = \frac{2}{\sqrt{\pi}} \int_X^\infty e^{-\beta^2} d\beta \quad \text{for} \quad 0 \leq X < \infty \quad \text{and} \quad F(0) = 1, \quad (13)$$

a function

$$u(r, t) = \int_0^t \left[F\left(\frac{ar}{2\sqrt{t-\eta}}\right) \frac{d\tau_0(\eta)}{d\eta} + F\left(\frac{a(R-r)}{2\sqrt{t-\eta}}\right) \frac{d\tau_1(\eta)}{d\eta} \right] d\eta \quad \text{for} \quad t \geq 0, \quad 0 \leq r \leq R \quad (14)$$

is defined which depends on the free functions $\tau_0(t)$ and $\tau_1(t)$. These functions are assumed to be continuous and continuously differentiable, and may be interpreted as heat sources (or sinks) at $r = 0$ and at $r = R$, respectively. The right-hand side of Equation 14 satisfies Equations 8 and 9, since $F(r, t)$ is a particular solution of Equation 8. The functions $\tau_0(t)$ and $\tau_1(t)$ are determined by substituting Equation 14 into Equations 11 and 12 and by specifying that $\tau_0(0) = \tau_1(0) = 0$. Nondimensional variables are defined by

$$t = \xi\sigma, \quad \tau_0(t) = T_0 R \omega_0(\sigma), \quad \text{and} \quad \tau_1(t) = T_0 R [-1 + \omega_1(\sigma)] \quad (15)$$

By use of Equations 11, 12, 14, and 15, the following system of nonlinear integro-differential equations for $\omega_0(\sigma)$ and $\omega_1(\sigma)$ is obtained:

$$\omega_0(\sigma) + \int_0^\sigma F\left(\frac{1}{2\sqrt{\sigma-\tau}}\right) \frac{d\omega_1(\tau)}{d\tau} d\tau = 0 \quad \text{for} \quad \sigma \geq 0, \quad (16)$$

and

$$\int_0^\sigma \left[\frac{d\omega_0(\tau)}{d\tau} \frac{\exp\left(\frac{-1}{4D^2(\sigma-\tau)}\right)}{\sqrt{\sigma-\tau}} - \frac{d\omega_1(\tau)}{d\tau} \frac{1}{\sqrt{\sigma-\tau}} \right] d\tau = \left[\int_0^\sigma F\left(\frac{1}{2D\sqrt{\sigma-\tau}}\right) \frac{d\omega_0(\tau)}{d\tau} d\tau + \omega_1(\sigma) \right]^4 -$$

$$- D\sqrt{\pi} \left[\omega_1(\sigma) - 1 + \int_0^\sigma F\left(\frac{1}{2D\sqrt{\sigma-\tau}}\right) \frac{d\omega_0(\tau)}{d\tau} d\tau \right] \text{ for } \sigma \geq 0 ,$$
(17)

where τ is a dummy variable for σ . Equations 14-17 confirm the above result of dimensional analysis.

Differentiation of Equation 16 yields $\omega_0^{(i)}(0) = 0$ for $i = 0, 1, 2, \dots$ and $0 < D < \infty$; i.e., the temperature changes initially only within an outer shell of the sphere under discussion.

For $D > 0$, the asymptotic expansion

$$\omega_1(\sigma) \sim 1 - \frac{2}{\pi} \sqrt{\sigma} + \left(\frac{4}{\pi} - \frac{D}{\sqrt{\pi}} \right) \sigma + \dots$$
(18)

is obtained. For $D = 0$ (i.e., $R = \infty$), the right-hand side yields a series expansion

$$\omega_1(\sigma) = 1 - (2/\pi) \sqrt{\sigma} + (4/\pi) \sigma + \dots$$
(18a)

in powers of $\sqrt{\sigma}$, whose circle of convergence has been determined approximately. The range of applicability of Equation 18 depends on D ; in case of $D = 0$, this range is given approximately by $0 < \sigma \leq 0.05$, according to numerical experience.

Equation (17) is converted by use of the well-known equivalence of

$$f(\chi) = \int_0^\chi \frac{g(z)}{\sqrt{\chi-z}} dz \quad \text{and} \quad \int_0^u g(z) dz = \frac{1}{\pi} \int_0^u \frac{f(\chi)}{\sqrt{u-\chi}} d\chi .$$
(18b)

The equation resulting from this process and Equation 16 have been solved by use of the initial conditions $\omega_0(0) = 0$ and $\omega_1(0) = 1$. The functions $\omega_0(\sigma)$ and $\omega_1(\sigma)$ were obtained at grid points $\sigma_i > 0$, $i = 1, 2, \dots$. For $0 < \sigma \ll 1$, the expansion in Equation 18 was used. The singularity at the upper limits $\tau = \sigma$ of the integrals was removed by use of an approximately valid closed-form evaluation of the integrals.

THE SEMI-INFINITE BODY ($R \rightarrow \infty$)

Because of Equation 6, the limiting case $R \rightarrow \infty$ is equivalent to $D \rightarrow 0$. By differentiating Equation 16 with respect to σ , it is seen that, for $D \rightarrow 0$, $d\omega_0(\sigma)/d\sigma$ tends to zero uniformly in σ within every finite interval on the positive σ -axis. By application of the equivalence (18b), Equation 17 becomes a nonlinear integral equation for $\omega_1(\sigma)$:

$$1 - \omega_1(\sigma) = \frac{1}{\pi} \int_0^\sigma \frac{\omega_1^4(\tau)}{\sqrt{\sigma - \tau}} d\tau, \text{ for } \sigma > 0, \text{ with } \omega_1(0) = 1. \quad (19)$$

Wolf and Mann (Reference 4) have shown both the existence and the uniqueness of solutions of the class of integral equations which includes Equation 19. According to this paper, the solution of Equation 19 satisfies the relations

$$0 < \omega_1(\sigma) \leq 1, \quad \omega_1'(\sigma) < 0, \text{ and } \lim_{\sigma \rightarrow \infty} \omega_1(\sigma) = 0. \quad (20)$$

The solution of Equation 19 has been obtained numerically using the expansion in Equation 18a.

A function $\omega_A(\sigma)$ is defined as a solution of

$$1 - \frac{1}{\pi} \int_0^\sigma \frac{\omega_A^4(\tau)}{\sqrt{\sigma - \tau}} d\tau = 0 \text{ for } \sigma > 0. \quad (21)$$

This equation is solved by

$$\omega_A(\sigma) = \sigma^{-1/8} \text{ for } \sigma > 0. \quad (22)$$

Because of Equations 20 and 21, $\sigma^{-1/8}$ may be interpreted as an approximately valid asymptotic solution of Equation 19 as $\sigma \rightarrow \infty$. If $\omega_1(\sigma)$ in the left-hand side of Equation 19 is equated to $\sigma^{-1/8}$ is of the order of $10^{-(n+1)}$ at $\sigma = \sigma_0 > 0$, the function $\omega_A(\sigma) = \sigma^{-1/8}$ satisfies Equation 19 with at least n decimals for $\sigma > \sigma_0$.

NUMERICAL RESULTS

The temperature $T(r, t)$ can be expressed in terms of the constants T_0 , ξ , and D and by use of the numerical solutions for $\omega_0(\sigma)$ and $\omega_1(\sigma)$. According to Equations 5 and 6, ξ and D depend on the initial temperature T_0 and geometrical and material properties which define the problem. In Figure 1, numerical results are presented for the time-dependence of temperatures at the surface and at certain stations inside spheres, with the radii

$$R = 1 \text{ cm}, \quad R = 15.3 \text{ cm}, \text{ and } R \rightarrow \infty, \quad (23)$$

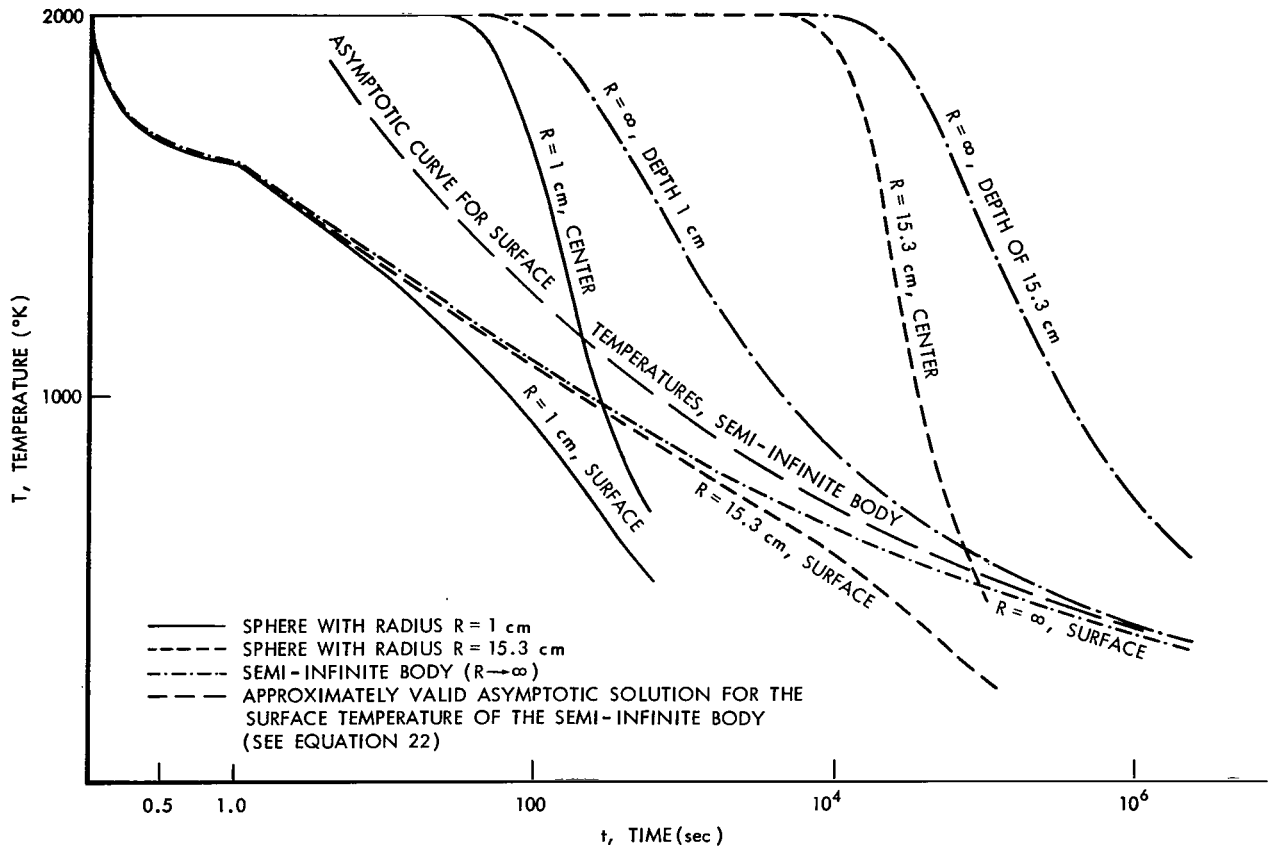


Figure 1—The curves with a vertical slope at $t = 0$ refer to surface temperatures; the other curves present temperatures at the center of the spheres or at the same distance from the surface of the semi-infinite body. The t -axis scale is linear over the range $0 < t < 1$ and logarithmic for $t > 1$ (sec).

the initial temperature

$$T_0 = 2000 \text{ } ^\circ\text{K} \quad , \quad (24)$$

and the material properties

$$K = 10^{-4} \text{ kcal/m } ^\circ\text{K sec} \quad ,$$

$$\rho = 2000 \text{ kg/m}^3 \quad ,$$

$$c = 0.3 \text{ kcal/kg } ^\circ\text{K} \quad ,$$

$$\epsilon_0 = 0.8 \quad ,$$

and

$$\sigma_0 \epsilon_0 = 1.1 \times 10^{-11} \text{ kcal/(m}^2 \text{ sec } ^\circ\text{K}^4) \quad . \quad (25)$$

The data of Equation 25 correspond approximately to the properties of quartz, which at $T_0 = 2000^\circ\text{K}$ is still almost opaque to thermal radiation. The sphere with $R = 15.3\text{ cm}$ weighs 30 kg.

Figure 1 presents the surface temperatures for spheres with the three radii as given in Equation 23, the temperatures at the centers of the two spheres with finite radii, the temperatures at the distances 1 cm and 15.3 cm underneath the surface of the semi-infinite body ($R \rightarrow \infty$), and the result of evaluating under the given conditions the approximately valid asymptotic solution (Equation 22) for the semi-infinite body.

The slope $\partial T(R, t)/\partial t$ of the surface temperature tends to $-\infty$ as time t tends to zero. This singularity is a consequence of prescribing a constant initial temperature T_0 and a finite heat transfer rate as $t \rightarrow 0$ (see Equations 2 and 4, respectively). In the solution derived, the singularity is caused by the complementary error integral (Equation 13) which exhibits different limits as $r \rightarrow R$ for $t = 0$, and as $t \rightarrow 0$ for $r = R$.

The temperature at a nonzero distance underneath the surface remains practically constant at the initial level $T = T_0$ for a time interval which increases together with this distance. Because of the converging cross-sectional areas as r decreases for finite R , the temperatures at the inner stations change more rapidly in the spheres of finite radius than in the semi-infinite body. With a small error increasing with time, the three surface temperatures presented in Figure 1 coincide until the temperatures at the centers of the spheres with finite radius R begin to drop. Initially, therefore, the semi-infinite solution furnishes a useful approximation for the surface temperatures of spheres with finite radius. In the semi-logarithmic presentation in Figure 1, the rates of decrease of all the temperatures presented are nearly linear with time for considerable intervals of time or temperature. An inspection of the graph suggests that the approximately valid asymptotic solution (Equation 22) yields a reasonable continuation of the computed surface temperature of the semi-infinite body.

The graph shows that the bodies made of quartz remain hot inside for a surprisingly long period of time. To check the order of magnitude of these results, a comparison to available linear solutions has been carried out by replacing Equation 4 by the linearized boundary condition

$$-K \frac{\partial T(R, t)}{\partial r} = \sigma_0 \epsilon_0 T_{\text{mean}}^3 T(R, t) \text{ for } t > 0, \quad (26)$$

where T_{mean} represents a suitably estimated constant number between the initial temperature T_0 and the surface temperature at the end of the time interval being considered. A heat transfer coefficient h is defined by

$$h = \sigma_0 \epsilon_0 T_{\text{mean}}^3 / K = \text{constant}. \quad (27)$$

The temperature history at the surface and at the center of spheres with finite radius R is presented as a function of T_0 , hR , and $h^2 t/a^2$ in Reference 2 and Reference 3, where a is defined in connection with Equation 1. Comparison with Figure 1 of data taken from either of these sources shows approximate coincidence of the time elapsing before the temperature begins to change at the centers

of the spheres with finite radii, and the average rate of decrease of the temperatures at these centers. This agreement obviously depends on a trial and error choice of T_{mean} in Equation 26, and thus does not permit us to use linearized solutions in the problem under discussion except for sufficiently small time intervals.

The numerical results presented pertain to constant material properties approximating those of quartz; these data show that internal heat conduction and emission of radiation from the surface furnish an inefficient way of cooling objects in space. If the cooling process begins at temperatures higher than 2000°K, internal radiative heat transfer will initially cause a rapid temperature decrease in the entire body; at a temperature close to 2000°K, however, internal radiation becomes insignificant in quartz, and the cooling process outlined in this paper takes over.

In this paper, the only heat transfer considered inside the sphere with radius R is that by conduction. Greenland and Lovering (Reference 5) have considered the complementary case when heat conduction is absent and the material is sufficiently transparent to thermal radiation as to render $T(r, t)$ a function of time t only. From their heat balance equation

$$-\rho c \frac{4}{3} \pi R^3 \frac{dT(t)}{dt} = 4\pi R^2 \sigma_0 \epsilon_0 T^4(t) \quad , \quad (28)$$

Greenland and Lovering obtain by straightforward integration the elementary closed-form solution

$$t = \frac{R\rho c}{9\sigma_0 \epsilon_0} [T(t)^{-3} - T(0)^{-3}] \quad . \quad (29)$$

Because of its independence of the radial coordinate, $T(r, t)$ as given by Equation 29 may be used to determine a temperature history prior to applying the solution derived in this paper, which rests on the assumption of a constant initial temperature $T(r, 0) = T_0 = \text{constant}$.

Goddard Space Flight Center
National Aeronautics and Space Administration
Greenbelt, Maryland, January 10, 1967
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REFERENCES

1. Carslaw, H. S. and Jaeger, J. C., "Conduction of Heat in Solids," Oxford: Clarendon Press, 1959.
2. Grigull, U., "Temperaturausgleich in einfachen Körpern: ebene Platte, Zylinder, Kugel, halbbunendlicher Körper," Berlin: Springer, 1964.

3. Gröber, H., Erk, S., and Grigull, U., "Die Grundgesetze der Wärmeübertragung," 3d ed., Berlin: Springer, 1963.
4. Mann, W. R. and Wolf, F., "Heat Transfer Between Solids and Gases Under Nonlinear Boundary Conditions," *Quart. Appl. Math.*, 9 (2): 163-184, July 1951.
5. Greenland, L. P. and Lovering, J. F., "Selective Volatilization from Tektites," *Nature*, 196(4860): 1195-1196, December 22, 1962.

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